向量的外積

$$\begin{cases} a_1x+a_2y+a_3z=0 & —① \\ b_1x+b_2y+b_3z=0 & —② \end{cases}$$

$$\begin{matrix} a_2 & & a_3 & & a_1 & & a_2 \\ & \times & & \times & & \times & \\ b_2 & & b_3 & & b_1 & & b_2 \end{matrix}$$

①$\times b_3$ − ②$\times a_3$ 消去 z 得

$$(a_1b_3-a_3b_1)x+(a_2b_3-a_3b_2)y=0$$

$$(a_2b_3-a_3b_2)y=(a_3b_1-a_1b_3)x$$

$$x:y=(a_2b_3-a_3b_2):(a_3b_1-a_1b_3)=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}:\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}$$

①$\times b_1$ − ②$\times a_1$ 消去 x 得

$$(a_2b_1-a_1b_2)y+(a_3b_1-a_1b_3)z=0$$

$$(a_3b_1-a_1b_3)z=(a_1b_2-a_2b_1)y$$

$$y:z=(a_3b_1-a_1b_3):(a_1b_2-a_2b_1)=\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}:\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

故 $x:y:z=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}:\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}:\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$\vec{n} = (x, y, z)$

$$\begin{matrix} a_2 & \times & a_3 & \times & a_1 & \times & a_2 \\ b_2 & & b_3 & & b_1 & & b_2 \end{matrix}$$

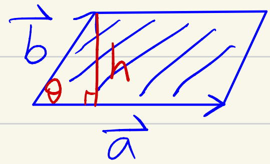
$$\begin{cases} \vec{a} \perp \vec{n} \\ \vec{b} \perp \vec{n} \end{cases} \rightarrow \begin{cases} \vec{a} \cdot \vec{n} = 0 \\ \vec{b} \cdot \vec{n} = 0 \end{cases} \rightarrow \begin{cases} a_1 x + a_2 y + a_3 z = 0 \\ b_1 x + b_2 y + b_3 z = 0 \end{cases}$$

得 $x : y : z = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} : \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} : \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

可取 $\vec{n} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$



$h = |\vec{b}|\sin\theta$

$$
\begin{aligned}
▱ &= |\vec{a}||\vec{b}|\sin\theta \\
&= |\vec{a}||\vec{b}|\sqrt{1-\cos^2\theta} \\
&= \sqrt{|\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta} \\
&= \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2} \\
&= \sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2}
\end{aligned}
$$

而 $(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) = a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2$ —①

$(a_1b_1+a_2b_2+a_3b_3)^2 = a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3$ —②

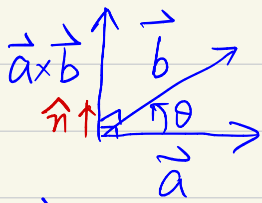
①－② 消去 $a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2$ 得

$(a_1b_2)^2 + (a_1b_3)^2 + (a_2b_1)^2 + (a_2b_3)^2 + (a_3b_1)^2 + (a_3b_2)^2 - 2(a_1b_2)(a_2b_1) - 2(a_1b_3)(a_3b_1) - 2(a_2b_3)(a_3b_2)$

可配成 $(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

即 $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$

故 $▱ = \sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2} = |\vec{a}||\vec{b}|\sin\theta$



$$\vec{a}=(a_1,a_2,a_3)$$
$$\vec{b}=(b_1,b_2,b_3)$$

$$\begin{cases}\vec{a}\perp(\vec{a}\times\vec{b})\\ \vec{b}\perp(\vec{a}\times\vec{b})\end{cases}\rightarrow \vec{a}\times\vec{b}=\left(\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix},\begin{vmatrix}a_3&a_1\\b_3&b_1\end{vmatrix},\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\right)$$

$$\because\ \text{▱}=|\vec{a}\times\vec{b}|=\sqrt{\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}^2+\begin{vmatrix}a_3&a_1\\b_3&b_1\end{vmatrix}^2+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}^2}$$

$$\therefore\ \vec{a}\times\vec{b}=|\vec{a}\times\vec{b}|\,\hat{n}=(|\vec{a}||\vec{b}|\sin\theta)\,\hat{n}$$

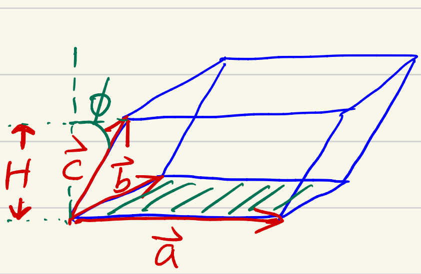
其中$\begin{cases}\vec{a}\perp\hat{n}\\ \vec{b}\perp\hat{n}\end{cases}$且遵守右手定則

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

降階法則

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \times \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \times \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \times \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$



$H = |\vec{c}||\cos\phi|$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$\vec{c} = (c_1, c_2, c_3)$

$\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$

$V_{\text{平行六面體}} =$  $\cdot H$

$= |\vec{a} \times \vec{b}| \cdot |\vec{c}||\cos\phi|$

$= \left| |\vec{a} \times \vec{b}||\vec{c}|\cos\phi \right|$

$= |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

$= \left| c_1 \times \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + c_2 \times \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + c_3 \times \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$

$= \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|$

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

※ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_1 \times \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \times \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \times \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$= c_1 \times \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + c_2 \times \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + c_3 \times \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$