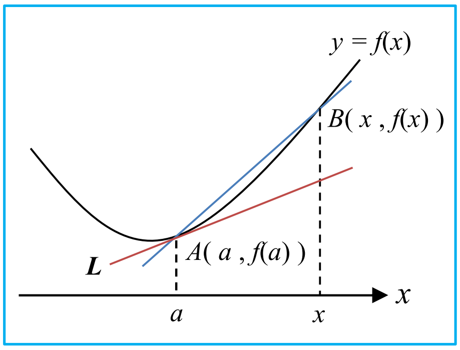
☆ 導數與導函數：



如右圖，割線 AB 之斜率為 $\dfrac{f(x)-f(a)}{x-a}$

切線 L 之斜率為 $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ 稱為 $y=f(x)$ 在 $x=a$ 處的導數

設 $x-a=h$，則 $x=a+h$ 且當 $x\to a$ 時 $h\to 0$

故 $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ (導數的另一形式)

當 a 值變動，與 $f'(a)$ 的值形成一函數關係，則將 a 換成 x，可得

導函數 $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

☆ 多項式函數的微分：

1. $f(x)=x^n\ (n\in N)\ \to\ f'(x)=nx^{n-1}$

$$
\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}\\
&=\lim_{h\to 0}\frac{C_0^n x^n+C_1^n x^{n-1}h+C_2^n x^{n-2}h^2+\cdots\cdots+C_n^n h^n-x^n}{h}\\
&=\lim_{h\to 0}\frac{\cancel{x^n}+nx^{n-1}h+(C_2^n x^{n-2}h^2+\cdots\cdots+C_n^n h^{n-2})h^2-\cancel{x^n}}{h}\\
&=\lim_{h\to 0}[nx^{n-1}+(C_2^n x^{n-2}h^2+\cdots\cdots+C_n^n h^{n-2})h]\\
&=nx^{n-1}
\end{aligned}
$$

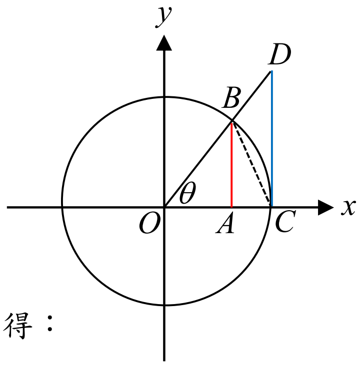
2. $f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}\ \to\ f'(x)=\dfrac{1}{3}x^{-\frac{2}{3}}$

$$
\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{(x+h)^{\frac{1}{3}}-x^{\frac{1}{3}}}{h}\\
&=\lim_{h\to 0}\frac{[(x+h)^{\frac{1}{3}}-x^{\frac{1}{3}}][(x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}]}{h\cdot[(x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}]}\\
&=\lim_{h\to 0}\frac{(x+h)-x}{h\cdot[(x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}]}\\
&=\frac{1}{x^{\frac{2}{3}}+x^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}}\\
&=\frac{1}{3}x^{-\frac{2}{3}}
\end{aligned}
$$

(亦符合 $f(x)=x^n\ \to\ f'(x)=nx^{n-1}$)

三角函數(Trigonometric function)微積分

☆ 三角函數的微分：



1. $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

如右圖($0 < \theta < \dfrac{\pi}{2}$)，圓 O 為單位圓

$\overline{AB} = \sin\theta$，$\overset{\frown}{BC} = \theta$，$\overline{CD} = \tan\theta$，作 $\overline{BC}$，則得：

ΔOBC 面積 $= \dfrac{1}{2} \times \overline{OC} \times \overline{AB} = \dfrac{1}{2} \times 1 \times \sin\theta = \dfrac{1}{2}\sin\theta$

扇形 OBC 面積 $= \dfrac{1}{2} \times 1^2 \times \theta = \dfrac{1}{2}\theta$

ΔOCD 面積 $= \dfrac{1}{2} \times \overline{OC} \times \overline{CD} = \dfrac{1}{2} \times 1 \times \tan\theta = \dfrac{1}{2}\tan\theta$

因為 ΔOBC 面積 < 扇形 OBC 面積 < ΔOCD 面積

所以 $\dfrac{1}{2}\sin\theta < \dfrac{1}{2}\theta < \dfrac{1}{2}\tan\theta \rightarrow \sin\theta < \theta < \tan\theta$

① $\sin\theta < \theta \rightarrow \dfrac{\sin\theta}{\theta} < 1$

② $\theta < \tan\theta \rightarrow \theta < \dfrac{\sin\theta}{\cos\theta} \rightarrow \cos\theta < \dfrac{\sin\theta}{\theta}$

故 $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

因為 $\lim\limits_{\theta \to 0}\cos\theta = \lim\limits_{\theta \to 0} 1 = 1$，所以由夾擠原理得知 $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

2. $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

$\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = \lim\limits_{\theta \to 0} \dfrac{(\cos\theta - 1)(\cos\theta + 1)}{\theta(\cos\theta + 1)} = \lim\limits_{\theta \to 0} \dfrac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim\limits_{\theta \to 0} \dfrac{-\sin^2\theta}{\theta(\cos\theta + 1)}$

由 $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ 得知當 $\theta \to 0$ 時 $\sin\theta \to \theta$

故 $\lim\limits_{\theta \to 0} \dfrac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \lim\limits_{\theta \to 0} \dfrac{-\theta^2}{\theta(\cos\theta + 1)} = \lim\limits_{\theta \to 0} \dfrac{-\theta}{(\cos\theta + 1)} = \dfrac{-0}{1+1} = 0$

得知 $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

3. $f(x)=\sin x \rightarrow f'(x)=\cos x$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= (\lim_{h\to 0}\frac{\sin h}{h})\cos x+(\lim_{h\to 0}\frac{\cos h-1}{h})\sin x \\
&= 1\cdot\cos x+0\cdot\sin x \\
&= \cos x
\end{aligned}
$$

4. $g(x)=\cos x \rightarrow g'(x)=-\sin x$

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= (\lim_{h\to 0}\frac{\sin h}{h})(-\sin x)+(\lim_{h\to 0}\frac{\cos h-1}{h})\cos x \\
&= 1\cdot(-\sin x)+0\cdot\cos x \\
&= -\sin x
\end{aligned}
$$

5. $h(x)=\tan x \rightarrow h'(x)=\sec^2 x$

$$
h(x)=\tan x=\frac{\sin x}{\cos x}
$$

$$
\begin{aligned}
h'(x) &= \frac{(\sin x)'\cdot\cos x-\sin x\cdot(\cos x)'}{\cos^2 x} \\
&= \frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}
$$

6. $k(x) = \cot x \quad \rightarrow \quad k'(x) = -\csc^2 x$

$$k(x) = \cot x = \frac{\cos x}{\sin x}$$

$$k'(x) = \frac{(\cos x)' \cdot \sin x - \cos x \cdot (\sin x)'}{\sin^2 x}$$

$$= \frac{(-\sin x) \cdot \sin x - \cos x \cdot (\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

$$= -\csc^2 x$$

7. $u(x) = \sec x \quad \rightarrow \quad u'(x) = \tan x \sec x$

$$u(x) = \sec x = \frac{1}{\cos x}$$

$$u'(x) = \frac{(1)' \cdot \cos x - 1 \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= (\frac{\sin x}{\cos x})(\frac{1}{\cos x})$$

$$= \tan x \sec x$$

8. $v(x) = \csc x \quad \rightarrow \quad v'(x) = -\cot x \csc x$

$$v(x) = \csc x = \frac{1}{\sin x}$$

$$v'(x) = \frac{(1)' \cdot \sin x - 1 \cdot (\sin x)'}{\sin^2 x}$$

$$= \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -(\frac{\cos x}{\sin x})(\frac{1}{\sin x})$$

$$= -\cot x \csc x$$

☆ 積分練習一：

1. $\int \cos x dx = \sin x + C$

2. $\int \cos^2 x dx = \int (\frac{1}{2} + \frac{1}{2}\cos 2x) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

3. $\int \cos^3 x dx = \int \cos^2 x \cdot \cos x dx = \int (1 - \sin^2 x) d(\sin x) = \sin x - \frac{1}{3}\sin^3 x + C$

4. $\int \cos^4 x dx = \int [\frac{1}{2}(1 + \cos 2x)]^2 dx$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}[x + \sin 2x + \frac{1}{2}\int (1 + \cos 4x) dx]$$

$$= \frac{1}{4}[x + \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x + k]$$

$$= \frac{1}{4}[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x + k]$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

5. $\int \cos^5 x dx = \int \cos^4 x \cdot \cos x dx$

$$= \int (1 - \sin^2 x)^2 d(\sin x)$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) d(\sin x)$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

6. $\int \cos^6 x dx = \int [\frac{1}{2}(1 + \cos 2x)]^3 dx$

$$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x) dx$$

$$= \frac{1}{8}[x + \frac{3}{2}\sin 2x + \frac{3}{2}\int (1 + \cos 4x) dx + \frac{1}{2}\int (1 - \sin^2 2x) d(\sin 2x)]$$

$$= \frac{1}{8}[x + \frac{3}{2}\sin 2x + \frac{3}{2}x + \frac{3}{8}\sin 4x + \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + k]$$

$$= \frac{1}{8}[\frac{5}{2}x + 2\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{6}\sin^3 2x + k]$$

$$= \frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

☆ e^x 與 $\ln(x)$ 的微分公式：

1. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\cdot\ln(1+x)=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln(\lim_{x\to 0}(1+x)^{\frac{1}{x}})=\ln(e)=1$$

※ 亦即 $\lim\limits_{x\to 0}\dfrac{x}{\ln(1+x)}=1$

2. $f(x)=e^x \;\to\; f'(x)=e^x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=(\lim_{h\to 0}\frac{e^h-1}{h})\cdot e^x$$

設 $e^h-1=y$，則 $e^h=1+y \;\to\; h=\ln(1+y)$

當 $h\to 0$ 時 $y\to 0 \;\to\; f'(x)=[\lim\limits_{y\to 0}\dfrac{y}{\ln(1+y)}]\cdot e^x=1\cdot e^x=e^x$

※ $\int e^x dx=e^x+C$

3. $g(x)=\ln(x) \;\to\; g'(x)=\dfrac{1}{x}$

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}=\lim_{h\to 0}\frac{\ln(\frac{x+h}{x})}{h}=\lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{h}$$

設 $\dfrac{h}{x}=y$，當 $h\to 0$ 時 $y\to 0$

則 $g'(x)=\lim\limits_{h\to 0}\dfrac{\ln(1+\frac{h}{x})}{\frac{h}{x}}\cdot\dfrac{1}{x}=[\lim\limits_{y\to 0}\dfrac{\ln(1+y)}{y}]\cdot\dfrac{1}{x}=1\cdot\dfrac{1}{x}=\dfrac{1}{x}$

※ $\int\dfrac{1}{x}dx=\ln|x|+C$

4. $y=a^x \;\to\; \ln y=x\ln a \;\to\; \dfrac{d}{dx}(\ln y)=\dfrac{1}{y}\cdot y'=\ln a \;\to\; y'=(\ln a)y=(\ln a)\cdot a^x$

5. $y=\log_a x=\dfrac{\ln x}{\ln a} \;\to\; y'=\dfrac{1}{\ln a}\cdot\dfrac{1}{x}$

☆ 積分練習二：

1. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln|\sec x| + C$

2. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

3. $$\begin{aligned}\int \tan^3 x dx &= \int (\sec^2 x - 1)\tan x dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \sec x d(\sec x) - \int \tan x dx \\ &= \frac{1}{2}\sec^2 x - \ln|\sec x| + C\end{aligned}$$

4. $$\begin{aligned}\int \tan^4 x dx &= \int (\sec^2 x - 1)\tan^2 x dx \\ &= \int \sec^2 x \tan^2 x dx - \int \tan^2 x dx \\ &= \int \tan^2 x d(\tan x) - \int \tan^2 x dx \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C\end{aligned}$$

5. $$\begin{aligned}\int \tan^5 x dx &= \int (\sec^2 x - 1)^2 \tan x dx \\ &= \int (\sec^4 x - 2\sec^2 x + 1)\tan x dx \\ &= \int \sec^4 x \tan x dx - 2\int \sec^2 x \tan x dx + \int \tan x dx \\ &= \int \sec^3 x d(\sec x) - 2\int (\tan^2 x + 1)\tan x dx + \int \tan x dx \\ &= \frac{1}{4}\sec^4 x - 2\int \tan^3 x dx - \int \tan x dx \\ &= \frac{1}{4}\sec^4 x - 2(\frac{1}{2}\sec^2 x - \ln|\sec x|) - \ln|\sec x| + \mathrm{C} \\ &= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\end{aligned}$$

6. $\int \sec x dx = \underline{\ln|\tan x + \sec x| + C}$

因為 $d(\tan x) = \sec^2 x dx$ 且 $d(\sec x) = \tan x \sec x dx$

所以 $d(\tan x + \sec x) = (\sec^2 x + \tan x \sec x)dx = \sec x(\tan x + \sec x)dx$

得 $\sec x dx = \frac{1}{\tan x + \sec x} d(\tan x + \sec x)$

故 $\int \sec x dx = \int \frac{1}{\tan x + \sec x} d(\tan x + \sec x) = \ln|\tan x + \sec x| + C$

☆ 反三角函數

1. 定義域與值域：

$y=\sin x\,(-1\le y\le 1)\ \rightarrow\ x=\sin^{-1}y\,(-\frac{\pi}{2}\le x\le\frac{\pi}{2})$

$y=\cos x\,(-1\le y\le 1)\ \rightarrow\ x=\cos^{-1}y\,(0\le x\le\pi)$

$y=\tan x\,(y\in R)\ \rightarrow\ x=\tan^{-1}y\,(-\frac{\pi}{2}<x<\frac{\pi}{2})$

$y=\cot x\,(y\in R)\ \rightarrow\ x=\cot^{-1}y\,(0<x<\pi)$

$y=\sec x\,(-\infty<y\le -1$ 或 $1\le y<\infty)\ \rightarrow\ x=\sec^{-1}y\,(0\le x\le\pi$ 但 $x\ne\frac{\pi}{2})$

$y=\csc x\,(-\infty<y\le -1$ 或 $1\le y<\infty)\ \rightarrow\ x=\csc^{-1}y\,(-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ 但 $x\ne 0)$

2. 相消公式：

$y=\sin^{-1}x\ \rightarrow\ \sin y=\sin(\sin^{-1}x)=x\ \rightarrow\ \sin^{-1}(\sin y)=\sin^{-1}x=y$

$y=\cos^{-1}x\ \rightarrow\ \cos y=\cos(\cos^{-1}x)=x\ \rightarrow\ \cos^{-1}(\cos y)=\cos^{-1}x=y$

$y=\tan^{-1}x\ \rightarrow\ \tan y=\tan(\tan^{-1}x)=x\ \rightarrow\ \tan^{-1}(\tan y)=\tan^{-1}x=y$

$y=\cot^{-1}x\ \rightarrow\ \cot y=\cot(\cot^{-1}x)=x\ \rightarrow\ \cot^{-1}(\cot y)=\cot^{-1}x=y$

$y=\sec^{-1}x\ \rightarrow\ \sec y=\sec(\sec^{-1}x)=x\ \rightarrow\ \sec^{-1}(\sec y)=\sec^{-1}x=y$

$y=\csc^{-1}x\ \rightarrow\ \csc y=\csc(\csc^{-1}x)=x\ \rightarrow\ \csc^{-1}(\csc y)=\csc^{-1}x=y$

3. 負角公式：

$\sin^{-1}(-x)=-\sin^{-1}x$ ； $\cos^{-1}(-x)=\pi-\cos^{-1}x$ ； $\tan^{-1}(-x)=-\tan^{-1}x$

※ $\cos\theta=x\ \rightarrow\ \theta=\cos^{-1}x$

$\cos(\pi-\theta)=-x\ \rightarrow\ \pi-\theta=\cos^{-1}(-x)\ \rightarrow\ \cos^{-1}(-x)=\pi-\cos^{-1}x$

4. 餘角公式：

$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

※ 設 $\sin\theta = \cos(\frac{\pi}{2}-\theta) = x$，則 $\theta = \sin^{-1}x$ 且 $\frac{\pi}{2}-\theta = \cos^{-1}x$

故 $\sin^{-1}x + \cos^{-1}x = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$

5. 反三角函數微分：

(1) $y = \sin^{-1}x\ (|x|<1) \to x = \sin y \to \frac{d}{dx}x = \frac{d}{dx}\sin y \to 1 = \cos y \cdot \frac{d}{dx}y$

$\frac{d}{dx}y = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} \to \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

(2) $y = \cos^{-1}x\ (|x|<1) \to x = \cos y \to \frac{d}{dx}x = \frac{d}{dx}\cos y \to 1 = -\sin y \cdot \frac{d}{dx}y$

$\frac{d}{dx}y = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} \to \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

(3) $y = \tan^{-1}x\ (x \in R) \to x = \tan y \to \frac{d}{dx}x = \frac{d}{dx}\tan y \to 1 = \sec^2 y \cdot \frac{d}{dx}y$

$\frac{d}{dx}y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} \to \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

(4) $y = \cot^{-1}x\ (x \in R) \to x = \cot y \to \frac{d}{dx}x = \frac{d}{dx}\cot y \to 1 = -\csc^2 y \cdot \frac{d}{dx}y$

$\frac{d}{dx}y = \frac{-1}{\csc^2 y} = \frac{-1}{1+\cot^2 y} \to \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

(5) $y = \sec^{-1}x\ (|x|>1) \to x = \sec y \to \frac{d}{dx}x = \frac{d}{dx}\sec y \to 1 = \tan y \sec y \cdot \frac{d}{dx}y$

$\frac{d}{dx}y = \frac{1}{\sec y \tan y} = \frac{1}{\sec y(\pm\sqrt{\sec^2 y - 1})} \to \frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

※ ($x>1$ 時 $\tan y = \sqrt{\sec^2 y - 1}$ ； $x<-1$ 時 $\tan y = -\sqrt{\sec^2 y - 1}$)

(6) $y = \csc^{-1}x\ (|x|>1) \to x = \csc y \to \frac{d}{dx}x = \frac{d}{dx}\csc y \to 1 = -\cot y \csc y \cdot \frac{d}{dx}y$

$\frac{d}{dx}y = \frac{-1}{\csc y \cot y} = \frac{-1}{\csc y(\pm\sqrt{\csc^2 y - 1})} \to \frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$

☆ 積分練習三：

1. $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx = \int \frac{1}{\sqrt{1-(e^{2x})^2}}d(e^{2x}) = \frac{1}{2}\sin^{-1}(e^{2x})+C$

2. $\int \frac{1}{e^{-x}+e^{x}}dx = \int \frac{e^{x}}{1+e^{2x}}dx = \int \frac{1}{1+(e^{x})^2}d(e^{x}) = \tan^{-1}(e^{x})+C$

3. $\int \frac{\cos x}{\sqrt{3+\cos^2 x}}dx = \int \frac{1}{\sqrt{4-\sin^2 x}}d(\sin x) = \int \frac{1}{\sqrt{1-(\frac{\sin x}{2})^2}}d(\frac{\sin x}{2}) = \sin^{-1}(\frac{\sin x}{2})+C$

4. $\int \frac{1}{x\sqrt{4x^2-9}}dx = \int \frac{1}{3x\sqrt{(\frac{2}{3}x)^2-1}}dx$

設 $\frac{2}{3}x = \sec\theta$ ，則 $\frac{2}{3}dx = \tan\theta\sec\theta d\theta$

原式 $= \int \frac{1}{3\cdot\frac{3}{2}\sec\theta\cdot\tan\theta}\frac{3}{2}\tan\theta\sec\theta d\theta = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\sec^{-1}(\frac{2}{3}x)+C$

5. $\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{4+(x+1)^2}dx = \frac{1}{4}\int \frac{1}{1+(\frac{x+1}{2})^2}dx = \frac{1}{2}\int \frac{1}{1+(\frac{x+1}{2})^2}d(\frac{x+1}{2})$

$= \frac{1}{2}\tan^{-1}(\frac{x+1}{2})+C$

6. $\int \frac{\tan^{-1}x}{1+x^2}dx = \int \tan^{-1}x\,d(\tan^{-1}x) = \frac{1}{2}(\tan^{-1}x)^2+C$

7. $\int \frac{1}{1+\sec^2 x}dx = \int (1-\frac{\sec^2 x}{1+\sec^2 x})dx = x - \int \frac{1}{2+\tan^2 x}d(\tan x) = x - \int \frac{\sqrt{2}}{2[1+(\frac{\tan x}{\sqrt{2}})^2]}d(\frac{\tan x}{\sqrt{2}})$

$= x - \frac{1}{\sqrt{2}}\tan^{-1}(\frac{\tan x}{\sqrt{2}})+C$

☆ 分部積分法

$$d(uv)=udv+vdu \ \rightarrow \ udv=d(uv)-vdu \ \rightarrow \ \int udv=\int d(uv)-\int vdu=uv-\int vdu$$

☆ 積分練習四：

1. $\int xe^x dx=\int xd(e^x)=xe^x-\int e^x dx=xe^x-e^x+C$

2. $\int x^2e^x dx=\int x^2d(e^x)=x^2e^x-2\int xe^x dx=x^2e^x-2(xe^x-e^x)+C=x^2e^x-2xe^x+2e^x+C$

3. $\int x^3e^x dx=\int x^3d(e^x)=x^3e^x-3\int x^2e^x dx=x^3e^x-3(x^2e^x-2xe^x+2e^x)+C$

$\quad =x^3e^x-3x^2e^x+6xe^x-6e^x+C$

4. $\int x\sin xdx=\int xd(-\cos x)=-x\cos x-\int(-\cos x)dx=-x\cos x+\sin x+C$

5. $\int x^2\sin xdx=\int x^2d(-\cos x)=-x^2\cos x-2\int(-\cos x)xdx=-x^2\cos x+2\int xd(\sin x)$

$\quad =-x^2\cos x+2(x\sin x-\int\sin xdx)=-x^2\cos x+2x\sin x+2\cos x+C$

6. $\int\ln xdx=x\ln x-\int xd(\ln x)=x\ln x-\int x\cdot\frac{1}{x}dx=x\ln x-\int dx=x\ln x-x+C$

7. $\int(\ln x)^2dx=x(\ln x)^2-\int x\cdot 2\ln x\cdot\frac{1}{x}dx=x(\ln x)^2-2\int\ln xdx$

$\quad =x(\ln x)^2-2(x\ln x-x)+C=x(\ln x)^2-2x\ln x+2x+C$

8. $\int x\ln xdx=\int\ln xd(\frac{1}{2}x^2)=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2\cdot\frac{1}{x}dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\int xdx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$

9. 試求：(1) $\int e^x\cos xdx$ (2) $\int e^x\sin xdx$

$\because \int e^x\sin xdx=\int\sin xd(e^x)=e^x\sin x-\int e^x\cos xdx \ \rightarrow \ \int e^x\cos xdx+\int e^x\sin xdx=e^x\sin x \ \cdots①$

$\int e^x\cos xdx=\int\cos xd(e^x)=e^x\cos x+\int e^x\sin xdx \ \rightarrow \ \int e^x\cos xdx-\int e^x\sin xdx=e^x\cos x \ \cdots②$

$\therefore$ $\frac{①+②}{2}$ 得 $\int e^x\cos xdx=\frac{1}{2}e^x(\sin x+\cos x)+C$ ； $\frac{①-②}{2}$ 得 $\int e^x\sin xdx=\frac{1}{2}e^x(\sin x-\cos x)+C$

10. $\int x\tan^{-1}xdx=\int\tan^{-1}xd(\frac{1}{2}x^2)=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int x^2d(\tan^{-1}x)$

$\quad =\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int(1-\frac{1}{1+x^2})dx$

$\quad =\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+C$